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Transient Surface Temperatures in Rocket Nozzles

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Nomenclature

- c = specific heat of solid, Btu/lb-°F
- d = half-thickness of node, in.
- erf = error function
- h = heat-transfer coefficient, convective plus radiant heating, Btu/sec-in.^{2}-°F
- k = thermal conductivity of solid, Btu/sec-in.^{2}-°F/in.
- M_n = eigenvalues of $M_n \tan M_n = N_{Bi}$
- N_{Bi} = Biot number, $N_{Bi} = h\delta/k$
- N_{Fo} = Fourier modulus, $N_{Fo} = \alpha\theta/\delta^2$
- n = number of eigenvalues
- T = temperature, °F
- α = thermal diffusivity, in.^{2}/sec
- δ = thickness of plate, in.
- θ = time, sec
- ρ = density, lb/in.^{3}

Subscripts

- e = equivalent of solid material
- g = gas
- i = initial value at $\theta = 0$
- w = surface of wall
- r = recovery temperature

Basic study of heat flux, stress, chemical reaction and physical action at the inside surface of a rocket nozzle wall requires a knowledge of the wall surface temperature. To calculate the transient temperature response of the inside surface of the rocket nozzle, one can use either a simple analytical solution limited to the flat plate geometry,[1] or a digital computer solution based on finite difference techniques.[2] Efficient use of the first method depends upon knowing the number of eigenvalues necessary to obtain an accurate answer; this paper presents a correlation which will enable the user to predict an optimum number of eigenvalues at any specified time of heating desired. It will be shown that for more complex studies, and complex nozzle geometries, the second method is useful and is just as accurate as the first, provided that the depth of the first thickness increment (or "node") adjacent to heating surface is properly chosen.

Received September 16, 1963; revision received January 24, 1964. This work has been sponsored by Advanced Research Project Agency under the Contract NOrd 16640.

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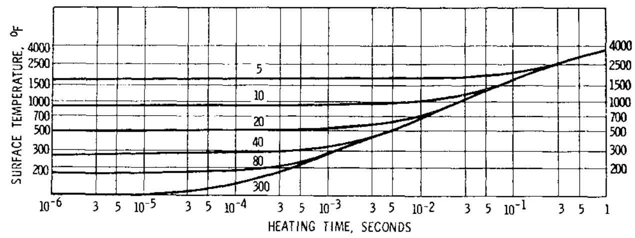

Fig. 1 Transient surface temperature for infinite flat plate calculated from Eq. (2) with varying number of eigenvalues.

Analytical Method (Flat Plate)

The equation for a semi-infinite solid,[1] with $\xi \equiv (\alpha\theta)^{1/2}h/k$, is

$$(T_w - T_i)/(T_r - T_i) = 1 - e^{\xi^2}[1 - \mathrm{erf}(\xi)] \qquad (1)$$

Equation (1) can be used only in flat-plate applications when the heating period (θ) is short, i.e., before the insulated back surface is heated. Typically, it is used for $\theta < 10^{-3}$ sec. For longer times, the equation for an infinite flat plate[1] should be used

$$\frac{T_w - T_r}{T_i - T_r} = 2 \sum_{n=1}^{n} \left(\frac{1}{1 + 2M_n/\sin 2M_n}\right) \exp(-M_n^2 N_{Fo}) \qquad (2)$$

In order to apply Eq. (2) correctly, a sufficient number of eigenvalues (which may become excessive for $\theta < 10^{-3}$ sec) must be used. For example, let $T_r = 6500$, $h = 0.03$, $k = 7 \times 10^{-4}$, $c = 0.3$, $\rho = 0.06$, $\delta = 1.5$, and $T_i = 70°$F (see nomenclature for units). For this case, Fig. 1 shows the T_w vs θ curves obtained with various numbers (n) of eigenvalues; for $\theta \geq 10^{-2}$ sec, $n = 40$ is sufficient, and for $\theta \geq 10^{-3}$, $n = 300$ is sufficient. The error for $\theta < 10^{-3}$ with $n = 300$ is seen, in Table 1, by comparing T_w's with those computed by Eq. (1).

To establish a method for determining the correct n to be used in Eq. (2), all parameters have to be generalized. For solid-propellant rockets, the common nozzle-insert materials are graphite (250°–7000°F) and molybdenum and tungsten (250°–5000°F).[3] Ranges of physical properties for these materials are:

$$3 \times 10^{-4} < k < 2 \times 10^{-3} \qquad 0.01 < c < 0.76$$
$$0.06 < \rho < 0.6 \qquad 10^4 < \alpha < 3 \times 10^5$$

Typical operating ranges for nozzle inserts are:

$$0.01 < h < 0.04 \qquad 0.1 < (T_w - T_r)/(T_i - T_r) < 1.0$$
$$0.125 < N_{Bi} < 33 \qquad 0.2 < N_{Fo} < 6.0$$

wherein the range of δ is assumed to be 0.3–3 in.

It has been found that practical correlations of the data are obtained by plotting θ vs $N_{Bi}\theta/N_{Fo}$ for various values of n, as in Fig. 2. The curves have been drawn through maximum time points to be conservative in estimating the minimum n for use in Eq. (2).

By appropriate use of Eqs. (1) and (2) as outlined previously, correct values for surface temperature can be obtained for any time period. These values then can be used to check surface temperatures obtained by a numerical method.

Numerical Method (Finite Differences)

Numerical methods (on computers) have obvious advantages, but their application to surface temperature calcu-

Table 1 Transient surface temperatures calculated by exact methods

	θ, sec	10^{-6}	10^{-5}	10^{-4}	10^{-3}	10^{-2}	10^{-1}
T_w, °F	Eq. (1)	76.0	90.0	130.0	256.8	638.6	1610.6
	Eq. (2)$_{n=300}$	100.0	103.7	132.5	257.2	638.7	1610.6

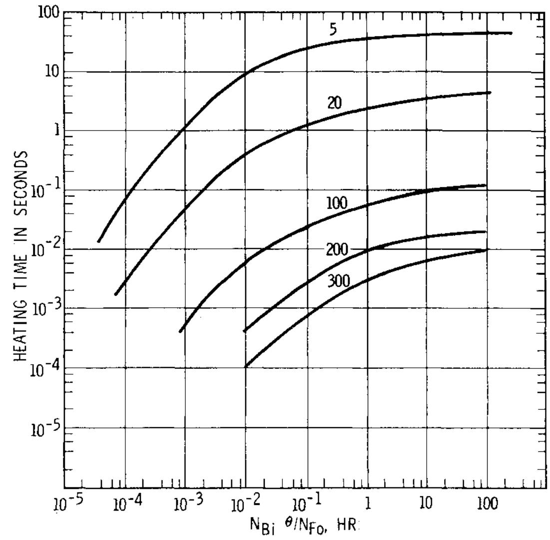

Fig. 2 Guidance for using Eq. (2).

lations has limitations due to truncation errors in time and space and convergence errors,[4, 5] particularly at the surface ($d = 0$) for short times ($\theta < 0.01$ sec). The numerical method used for the following calculations is the implicit or "backward-time-step" method, such as "Hetran,"[2] programmed on a digital computer using a Gauss-Seidell iteration for converging the system of equations.

The calculation of surface temperature by the linearization of temperature obtained by the numerical method is illustrated in Fig. 3, in which $\delta_e = k/h$, the hypothetical thickness of the plate, is analogous to δ_g, the thickness of the gas film. By linearization at a particular θ, the correct surface temperature T_w is obtained only with the optimum adjacent node thickness ($2d$). Otherwise, too thick a node ($2d'$) will give a high surface temperature, and too thin a node ($2d''$) will give a low surface temperature.

There is one distance d (the midpoint of the adjacent node) behind the heating surface for which the linearized surface temperature will be correct for a specific time, θ. Since it is not practical to use a variable d, the problem is to select the best d, i.e., one which will give an error $\leq 1\%$, for the θ of interest. This concept is tested in Fig. 4 by using various d's for the example studied previously by the analytical method. (Note the similarity between Figs. 1 and 4.) The transient surface temperature curves diverge widely at small times because of truncation errors, but for $2d \geq 10^{-3}$ in., they gradually come together during the transient to 40 sec. For $2d = 10^{-4}$ in., the surface temperatures remain erroneously low because of faulty iterative convergence and round-off error. A reasonable guide to avoid such convergence errors for successive time steps ($\Delta\theta$) is to keep $2d/\Delta\theta > 10$. Whenever the number of iterations for any given time step exceeds 8, the possibility of serious convergence error exists.

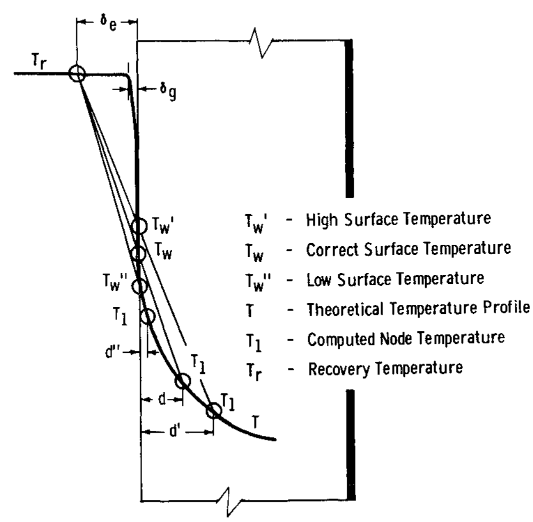

Fig. 3 The concept of surface temperature determined by linearization.

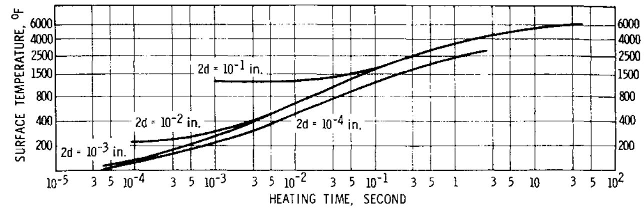

Fig. 4 Transient surface temperatures calculated from Hetran program with different thickness of node adjacent to heating surface in illustration sample.

The success of correlation between $N_{Bi}\ \theta/N_{Fo}$ and θ for various numbers of eigenvalues for Eq. (2) suggests that similar correlations might be useful in the numerical method. This was found to be the case for determining the lower-time limit for $2d = 0.1$ in. However, for $\theta < 0.1$ sec, the insulated surface does not experience a change in temperature; therefore δ loses significance and can be dropped from $N_{Bi}\ \theta/N_{Fo}$, leaving $h/k\alpha$ as the correlating parameter for $2d = 10^{-2}$ and 10^{-3} in. (Fig. 5). (For $\theta < 10^{-2}$ sec, α is of greater importance than either h or k, so that $h/k\alpha^2$ becomes a better correlating parameter.) The time steps used for obtaining the upper 1% limit as shown in Fig. 5 were $\Delta\theta = 10^{-4}$ sec for $10^{-4} < \theta < 10^{-3}$ sec, $\Delta\theta = 10^{-3}$ sec for the middle range up to 0.5 sec, and $\Delta\theta = 0.002$ sec for $0.5 < \theta < 5$ sec.

Based on the preceding study, three orders of node thickness ($2d$) are sufficient for most T_w calculations for heating times from 10^{-4} sec to more than 100 sec. At $\theta < 10^{-3}$ sec, $2d = 0.001$ in. is recommended; temperatures for such

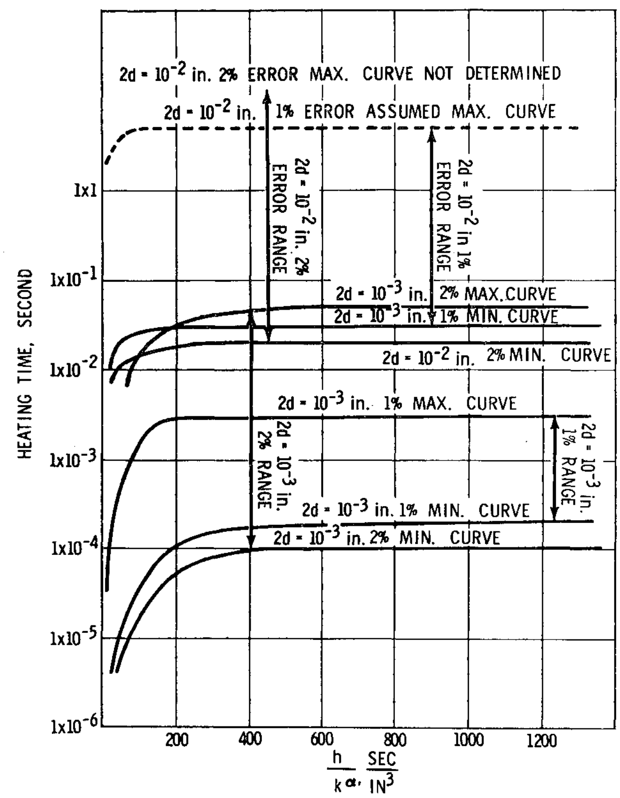

Fig. 5 Sketch for selection of node thickness adjacent to heating surface in numerical method for correct surface temperature.

short times are low ($\leq 150°F$), and little difference is seen between a 1% and a 2% error curve. For $\theta > 0.03$ sec, $2d = 0.01$ in. is appropriate. For the intermediate range, $0.001 < \theta < 0.03$ sec, either a 0.001-in. node can be used at an error between 1% and 2% ($\sim 5°F$), or a node thickness between 0.001 and 0.01 in. can be used to hold the error near 2%. For $\theta > 5$ sec, a 0.1-in.-node thickness is appropriate.

Conclusions

Transient surface temperatures T_w for a flat plate can be calculated by an analytical method within chosen confidence limits, provided that a sufficient number of eigenvalues n are taken; correlations of N_{Bi}/N_{Fo} vs θ are presented to aid in the selection of the minimum n. A numerical method is useful for the more complex problems (variable material properties and boundary conditions) encountered in practice; correlations for predicting the correct node thickness to give an accurate T_w for a desired θ by this approach are based on $N_{Bi}\ \theta/N_{Fo}$, $h/k\alpha$, and $h/k\alpha^2$ for the large, short, and very short times, respectively.

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Theoretical vs Actual Nike-Apache Sounding Rocket Performance

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THE Nike-Apache vehicle (see Fig. 1) is a two-stage unguided sounding rocket, which has been recently added to the NASA family of sounding rockets. It is described in detail in Refs. 1 and 2. Nike-Apache, NASA No. 14.108 GI, was fired at Wallops Island, Virginia, on March 9, 1963. The vehicle was equipped with telemetry turnstile antennas and carried a 76-lb payload. The launcher

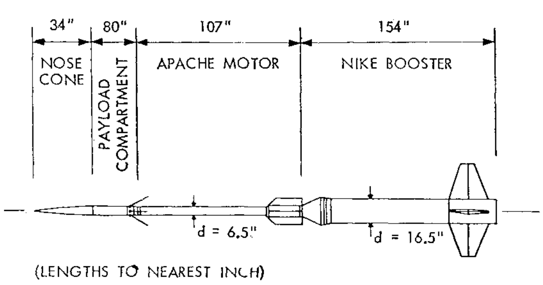

Fig. 1 Nike Apache.

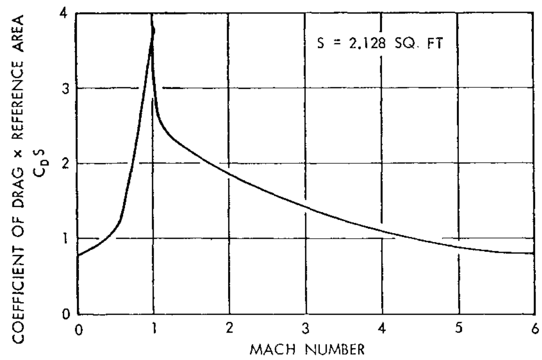

Fig. 2 C_DS vs Mach number, Nike, burning.

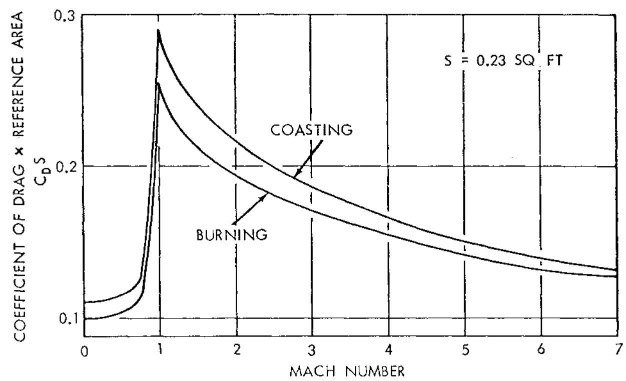

Fig. 3 C_DS vs Mach number, Apache with turnstile antenna, burning, and coasting.

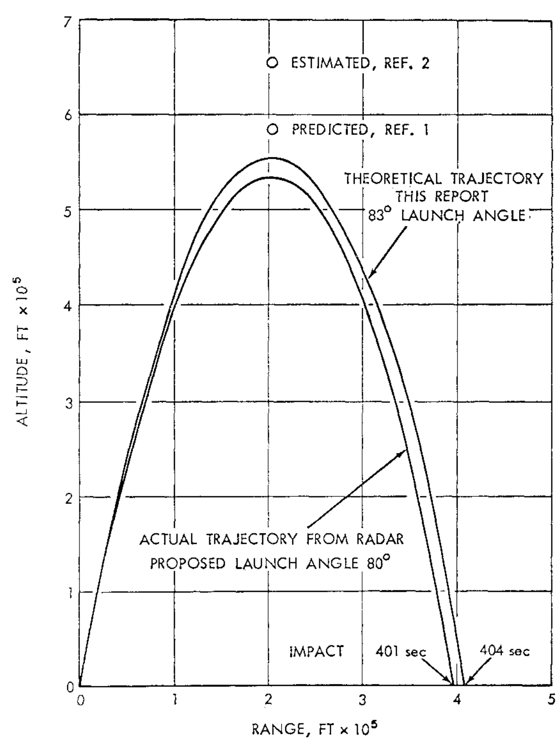

Fig. 4 Nike-Apache (14.108) trajectory.

was set at 108° azimuth and 75.7° elevation to correct for wind, the proposed launch angle being 80°.

The actual trajectory, as determined by smoothing measurements from two separate radar plots, was compared with theoretical trajectories computed for the Nike-Apache using the given payload, weights, thrusts, and drag conditions. This comparison indicates that the effective launch angle was slightly more than 83°. Corrections were made for the distance between the radar sites but not for the distance from

Received December 12, 1963; revision received January 17, 1964.

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